Solutions

Exam 1
Chapters 3 and 4

Answer the following questions. *You must show your work to receive full credit.* Be sure to make reasonable simplifications. If your answer includes Permutations or Combinations, please find the number it represents. For instance, $C(3,1) = 3$. Indicate your final answer with a $\boxed{\text{box}}$.

1. (10 points) Solve the following system of equations using matrices. You can use your calculator, but show all appropriate work for full credit.

$$\begin{array}{rcrcrcl} -\frac{1}{2}x & + & y & - & \frac{1}{2}z & = & 0 \\ -\frac{1}{2}x & - & \frac{1}{2}y & + & z & = & 0 \\ x & - & \frac{1}{2}y & - & \frac{1}{2}z & = & 0 \end{array}$$

$$\begin{pmatrix} -\frac{1}{2} & 1 & -\frac{1}{2} & 0 \\ -\frac{1}{2} & -\frac{1}{2} & 1 & 0 \\ 1 & -\frac{1}{2} & -\frac{1}{2} & 0 \end{pmatrix} \xrightarrow{RREF} \begin{pmatrix} 1 & 0 & -1 & 0 \\ 0 & 1 & -1 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}$$

Infinitely many solution. z is arbitrary

$$\begin{array}{l} x - z = 0 \\ y - z = 0 \end{array} \Rightarrow \begin{array}{l} x = z \\ y = z. \end{array} \quad \text{So} \quad (x,y,z) = (z,z,z).$$

2. (10 points) You own a hamburger franchise and are planning to shut down operations for the day, but you are left with 13 bread rolls, 19 defrosted beef patties, and 15 opened cheese slices. Rather than throw them out, you decide to use them to make burgers that you will sell at a discount. Plain burgers require 1 beef patty and 1 bread roll, double cheeseburgers require 2 beef patties, 1 bread roll and 2 slices of cheese, while regular cheeseburgers require 1 beef patty, 1 bread roll, and 1 slice of cheese. How many of each should you make if you want to use all your resources?

	Plain	Double	Regular	Total
Bread	1	1	1	13
Patties	1	2	1	19
Cheese	0	2	1	15

$$\begin{pmatrix} 1 & 1 & 1 & 13 \\ 1 & 2 & 1 & 19 \\ 0 & 2 & 1 & 15 \end{pmatrix} \xrightarrow{\text{RREF}} \begin{pmatrix} 1 & 0 & 0 & 4 \\ 0 & 1 & 0 & 6 \\ 0 & 0 & 1 & 3 \end{pmatrix}$$

<u>Unique Solution:</u> $(x, y, z) = (4, 6, 3)$.

You should make 4 plain hamburgers,
6 double cheeseburgers, and
3 regular cheeseburgers.

3. (5 points) Rewrite the following matrix equation as a system of linear equations.

$$\begin{bmatrix} 2 & -1 & 3 & -7 \\ 0 & 0 & 1 & -1 \\ 10 & -2 & 8 & -4 \end{bmatrix} \begin{bmatrix} x \\ y \\ z \\ w \end{bmatrix} = \begin{bmatrix} 1 \\ 1 \\ 1 \\ \cancel{1} \end{bmatrix}.$$

$$2x - y + 3z - 7w = 1$$

$$z - w = 1$$

$$10x - 2y + 8z - 4w = 1$$

4. (2 points each) Let A be a (2×9) matrix, B a (10×9) matrix. C a (89×10) matrix, and D a (9×89) matrix. Which of the following operations is defined?

(a) $A + A$ Defined $= 2A$, a 2×9 matrix.

(b) $B + A$ Undefined

(c) $B \cdot A$ Undefined

(d) $C \cdot D$ Undefined

(e) $B \cdot D \cdot C$ Defined, a 10×10 matrix.

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5. (5 points) Is the following matrix invertible? If so, what is the inverse matrix?

$$\begin{bmatrix} 1 & 1 & 1 \\ -1 & 1 & -1 \\ 0 & -1 & 1 \end{bmatrix}$$

$$\left(\begin{array}{ccc|ccc} 1 & 1 & 1 & 1 & 0 & 0 \\ -1 & 1 & -1 & 0 & 1 & 0 \\ 0 & -1 & 1 & 0 & 0 & 1 \end{array}\right) \xrightarrow{\text{RREF}} \left(\begin{array}{ccc|ccc} 1 & 0 & 0 & 0 & -1 & -1 \\ 0 & 1 & 0 & 1/2 & 1/2 & 0 \\ 0 & 0 & 1 & 1/2 & 1/2 & 1 \end{array}\right)$$

$$\text{Inverse matrix} = \begin{pmatrix} 0 & -1 & -1 \\ 1/2 & 1/2 & 0 \\ 1/2 & 1/2 & 1 \end{pmatrix}$$

6. You and your friend have come up with the following simple game to pass the time: at each round, you simultaneously call "heads" or "tails." If you both call "heads" your friend wins 2 points; if you both call "tails" your friend wins 1 points; if your call differs, then you win 2 points if you call "heads," and 1 point if you called "tails."

(a) (4 points) Set up the payoff matrix for this game. Be sure to label the moves and which player is which.

(b) (6 points) What is your optimal strategy and what is the expected payoff if you use this strategy?

(a)

Your friend

$$\text{You} \quad \begin{array}{c|cc} & H & T \\ \hline H & \multicolumn{2}{c}{\multirow{2}{*}{$\begin{pmatrix} -2 & 2 \\ 1 & -1 \end{pmatrix}$}} \\ T & & \end{array}$$

(b) $E_1 = r P_{c_1} = (x \;\; 1-x)\begin{pmatrix} -2 & 2 \\ 1 & -1 \end{pmatrix}\begin{pmatrix} 1 \\ 0 \end{pmatrix} = [1-3x]$

$E_2 = r P_{c_2} = (x \;\; 1-x)\begin{pmatrix} -2 & 2 \\ 1 & -1 \end{pmatrix}\begin{pmatrix} 0 \\ 1 \end{pmatrix} = [3x-1]$

$E_1 = E_2$ when $x = \frac{1}{3}$. When $x = \frac{1}{3}$, $E_1 = 0$.

So $r^* = (\frac{1}{3} \;\; \frac{2}{3})$ and the expected payoff is 0.

7. (10 points) A farmer has a choice of growing wheat, barley, or rice. Her success will depend on the weather, which could be dry, average, or wet. Her payoff matrix is as follows.

		Weather		
		Dry	Average	Wet
	Wheat	20	20	10
Crop Choices	Barley	10	15	20
	Rice	10	20	20

If the probability that the weather will be dry is 10%, the probability that it will be average is 60%, and the probability that it will be wet is 30%, what is the farmer's best choice of crop?

$$E = rPc = \begin{pmatrix} x & y & z \end{pmatrix} \begin{pmatrix} 20 & 20 & 10 \\ 10 & 15 & 20 \\ 10 & 20 & 20 \end{pmatrix} \begin{pmatrix} .1 \\ .6 \\ .3 \end{pmatrix}$$

$$= \begin{pmatrix} x & y & z \end{pmatrix} \begin{pmatrix} 17 \\ 16 \\ 19 \end{pmatrix} = (17x + 16y + 19z).$$

Maximized when $z=1$. Optimal strategy $= r^* = (0 \ 0 \ 1)$.

The farmer should grow rice.